A deficient spline function approximation for boundary layer flow

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Abstract *In this paper a spline approximation of deficiency 3 and a step of length 3h method is proposed to approximate the solution of the problem and its derivatives. The Falkner- Skan equation has been solved through the use of the shooting technique for handling the problem when the conditions imposed are of boundary-value rather than an initial-value type for different values of its parameters. Comparisons are made between the data resulting from the proposed method and those obtained by others.*

Introduction

The problem of laminar boundary layer resulting from the flow of an incompressible fluid past a semi-infinite wedge is of considerable practical and theoretical interest. Non-linear problems with semi-infinite domains are frequently encountered in the study of laminar boundary layer. Owing to the appearance of irregular boundaries, shock waves, boundary layers, derivative boundary conditions, etc., the solutions so obtained have in many cases been unsatisfactory because of poor resolution, spurious oscillations, and excessive computer time storage.

For the past decades, finite differences have been used extensively for the evaluation of flow mechanics. The similarity solution has been well studied for the testing of finite-difference methods. Beckett (1983), Rosenhead (1963), Wadia and Payne (1981) and White (1974) have discussed different finite difference schemes for the solution of the Falkner-Skan equation.

El-Gindy *et al.* (1995) use the Chebyshev spectral method with a modified Rayleigh-Ritz method to solve the Falkner-Skan equation. They obtain agreement results with the classical solutions.

In any case the far field boundary condition is a problem, where the correct value of unknown shear stress f''_w at the wall must be found, which ensures an asymptotic approach of the velocity values of f' at infinity to unity (the well-known matching condition of viscous solution (near-wall) to the inviscid solution). Such techniques are termed the "shooting method".

Thomas and Harris (1984) have studied the solution of the Falkner-Skan equation by a pseudo-spectral method for different values of the similarity parameter β. They have chosen a class of problems in which $\gamma = 1$. The

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Falkner-Skan equation was chosen as the equation of interest because of the inherent non-linearity that it exhibits. The equation is derived by White (1974), and in other textbooks on boundary layer theory.

We propose to approximate the solution of the problem and its derivatives using a spline approximation of deficiency 3 and a step of length 3h. The Falkner-Skan equation has been solved through the use of the shooting technique for handling the problem when the conditions imposed are of a boundary-value rather than an initial-value type for different values of its parameters. Comparisons are made between the data resulting from the proposed method and those obtained by El-Gindy *et al.* (1995), El-Hawary (1990), Rosenhead (1963), Wadia and Payne (1981), White (1974) and Beckett (1983).

Method of solution

Consider the Falkner-Skan equation flows with the similarity property (Rosenhead (1963) and White (1974)):

$$f'''(\eta) + \alpha f(\eta) f''(\eta) + \beta\left[1 - (f'(\eta))^2\right] = 0 \tag{1}$$

together with the boundary conditions

$$f(0) = f'(0) = 0,$$

and

$$f'(\eta) \to 1$$

as

$$\eta \to \infty.$$

Here α is assumed constant, β is a measure of the pressure gradient. The prime (′) denotes differentiation with respect to η. The special case of the Blasius similarity relation for incompressible viscous flow along a flat plate results where $\alpha = 1$ and $\beta = 0$.

The domain is $0 \leq \eta \leq \eta_\infty$ where η_∞ is one end of the user specified computational domain.

Sallam and El-Hawary (1983; 1984) considered a spline approximation, $S(x) \in C^{m-3}$ of deficiency 3 and a step of length H = 3h to the system

$$y' = f(x, y)\ , \qquad y(0) = y_0 \tag{2}$$

and satisfies the Lipschitz condition $f \in C^m$ in some domain.

$$D, D = \{(x, y) | 0 \leq x \leq b\}$$

They considered a spline $S(x)$ of deficiency 3 and a step of length $H = 3h$; $h = b/3N$; $N > m$.

Let $y(x)$ be the exact solution of equation (2) and $S(x)$ its spline approximation on $I_n = [nH, (n+1)H], n = 0(1)N-1$, where:

$$S(x) = \sum_{i=0}^{m-3} \frac{(x-3nh)^i}{i!} S_{3n}^{(i)} + \sum_{i=m-2}^{m} \frac{(x-3nh)^i}{i!} C_{i,n}$$

and the coefficient vectors to be determined by the conditions:

$$S'(jh) = f(jh,\ S(jh)),\ j = 3n+1,\ 3n+2, 3n+3$$

The above construction is uniquely defined vector-valued spline $S(x) \in C^{m-3}$ whenever

$$0 \leq h \leq h_0 = \frac{m-2}{L(4m-5)(m+2)}$$

where L is Lipschitz constant.

They have shown that the method is unstable and hence divergent for $m \geq 6$. The deficient vector-valued spline function approximation are A-stable for $m = 4$ (Sallam and El-Hawary, 1983).

The convergence of the spline approximation of deficiency 3 to systems of first-order differential equations is for $m = 4$ and $m = 5$ (Sallam and El-Hawary, 1984). In addition, global error bounds of the form

$$\left\|S^{(i)} - y^{(i)}(x)\right\|_\infty = O\left(h^{m+1-i}\right) \quad , \quad i = 0(1)m.$$

For finding an approximate solution to differential equation (1), we need to know $f''(0)$.We shall use the technique for handling the problem when the conditions imposed are of a boundary-value rather than an initial-value type.

We can put equation (1) in the form:

$$f'''(\eta,t) + \alpha f(\eta,t) f'(\eta,t) + \beta\left[1 - (f'(\eta,t))^2\right] = 0 \qquad \text{(3a)}$$

$$f(0,t) = f'(0,t) = 0,\ \text{and}\, f'(0,t) = t \qquad \text{(3b)}$$

We do this by choosing the parameters in a manner to ensure that

$$\lim_{k\to\infty} f(\eta_\infty, t_k) = f(\eta_\infty) = 1$$

where $f(\eta, t_k)$ denotes the solution of the boundary-value problem (3) with $t = t_k$ and $f(\eta)$ denotes the solution to the boundary value problem (1).

We start with a parameter t_0 that determines the initial elevation at which the object is fired from the point (0,0) and along the curve described by the solution to the initial-value problem:

$$f'''(\eta) + \alpha f(\eta)f''(\eta) + \beta\left[1 - (f'(\eta))^2\right] = 0 \\ f(0) = f'(0) = 0, \text{ and } f''(0) = t_0 \tag{4}$$

If $f'(\eta_\infty, t_0)$ is not sufficiently close to 1, we attempt to correct our approximation by choosing another elevation t_1 and so on, until $f'(\eta_\infty, t_k)$ is sufficiently close to 1.

To determine the parameter t_k, suppose a boundary-value problem of the form (1). If $f(\eta, t)$ denotes the solution to the initial-value problem (4), the problem is to determine t so that:

$$f(\eta_\infty, t) - 1 = 0$$

We shall use Newton's method to generate the sequence $\{t_k\}$; only one initial value, t_0, is needed, however: the iteration has the form:

$$t_k = t_{k-1} - \frac{f'(\eta_\infty, t_{k-1}) - 1}{\frac{\partial f'}{\partial t}(\eta_\infty, t_{k-1})} \tag{5}$$

and requires knowledge of $\frac{\partial f'}{\partial t}(\eta_\infty, t_{k-1})$.

This presents some difficulty, since an explicit representation for: $f'(\eta_\infty, t)$ is not known; we know only the values $f'(\eta_\infty, t_0), f'(\eta_\infty, t_1), \ldots, f'(\eta_\infty, t_{k-1})$.

We take the partial derivative of problem (3). This implies that:

$$\frac{\partial}{\partial t} f'''(\eta, t) = \frac{\partial}{\partial \eta}[-\alpha ff'' - \beta(1 - f'^2)]\frac{d\eta}{dt} + \frac{\partial}{\partial f}[-\alpha ff'' - \beta(1 - f'^2)]\frac{\partial f}{\partial t} \\ + \frac{\partial}{\partial f'}[-\alpha ff'' - \beta(1 - f'^2)]\frac{\partial f'}{\partial t} + \frac{\partial}{\partial f''}[-\alpha ff'' - \beta(1 - f'^2)]\frac{\partial f''}{\partial t}$$

Table I.
The wall shear stress f''_w of present method, $m = 5$

η_∞	α	β	f''	No. of iterations	h
8	1	–0.15	–0.1334299	5	$\frac{1}{45}$
8	1	–0.18	–0.0976953	5	$\frac{1}{51}$
5.6	1	–0.18	0.1289903	4	$\frac{1}{45}$
5.2	1	–0.15	0.2167541	4	$\frac{1}{45}$
6.9	1	0	0.4696000	4	$\frac{1}{48}$
4.4	1	0.30	0.7747827	4	$\frac{1}{51}$
3.7	1	0.5	0.9278054	4	$\frac{1}{45}$
3.5	1	1	1.2326171	5	$\frac{1}{45}$
3.1	1	2	1.6872256	5	$\frac{1}{54}$
4.6	0	1	1.1547118	4	$\frac{1}{45}$
2	1	10	3.675213	5	$\frac{1}{45}$
2	1	15	4.4914630	5	$\frac{1}{45}$

Since η and t are independent, then

$$\frac{\partial}{\partial t} f'''(\eta, t) = -\alpha f'' \frac{\partial f}{\partial t} + 2\beta f' \frac{\partial f'}{\partial t} - \alpha f \frac{\partial f''}{\partial t}, \quad \eta \in [0, \eta_\infty] \tag{6}$$

The initial conditions give

$$\frac{\partial}{\partial t} f(0, t) = 0, \quad \frac{\partial}{\partial t} f'(0, t) = 0 \quad \text{and} \quad \frac{\partial f''}{\partial t}(0, t) = 1$$

η_∞	α	β	(El-Hawary, 1990)	(El-Gindy *et al.*, 1995)	(Wadia and Payne, 1981)	(Rosenhead, 1963)	(White, 1974)	(Beckett, 1983)	Present method
2	1	15	4.4914	4.4905	3.9316			4.4923	4.4916430
2	1	10	3.6752	3.6746	3.33			3.6756	3.675213
3.1	1	2	1.6712	1.7741	1.6866	1.6866	1.6872	1.6874	1.6872256
3.7	1	0.5	0.929	0.694	0.926	0.9281		0.92778	0.9278054
4.4	1	0.3	0.7768	0.5332	0.7747		0.7748		0.7747827

Table II. Comparison of the wall shear stress

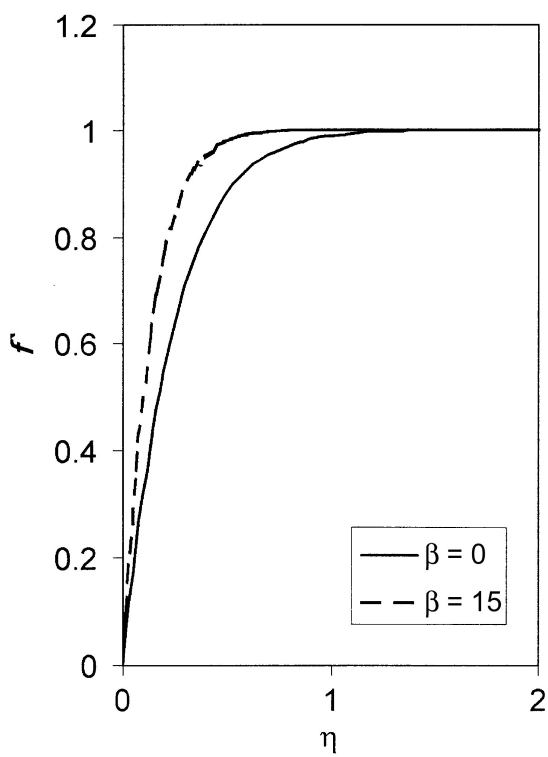

Figure 1. Spline approximation method of Falkner-Skan equation for different values of $\beta, \alpha = 1, \eta_\infty = 2$

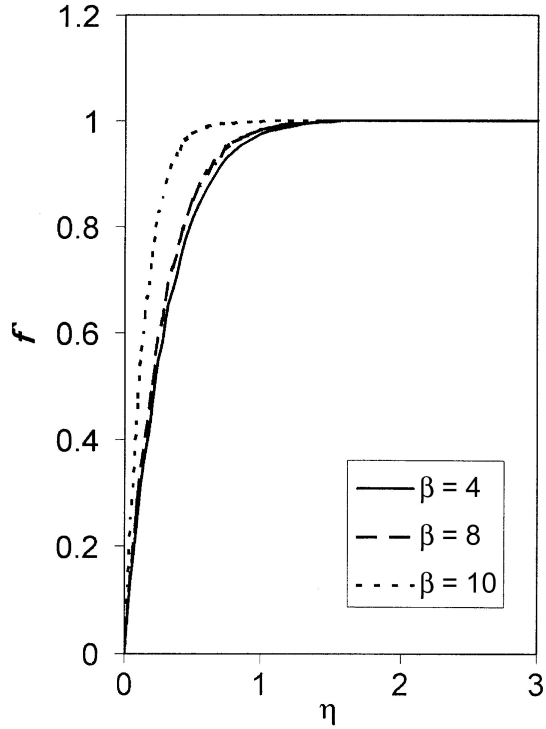

Figure 2.
Spline approximation method of Falkner-Skan equation for different values of $\beta, \alpha = 1$, $\eta_{\infty} = 3$

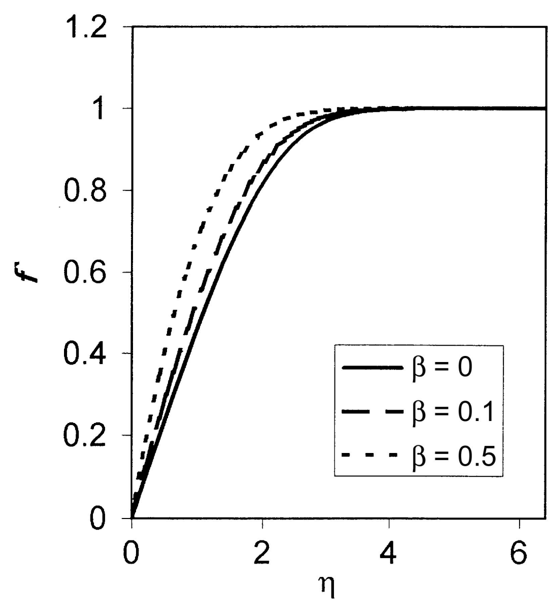

Figure 3.
Spline approximation method of Falkner-Skan equation for different values of $\beta, \alpha = 1$, $\eta_{\infty} = 6.4$

By using $z(\eta, t)$ to denote $\frac{\partial}{\partial t} f(\eta, t)$ and assume that the order of differentiation of η and t can be reversed, we have the initial-value problem

$$\begin{aligned} & z'''(\eta, t) = -\alpha f''(\eta, t) z(\eta, t) + 2\beta f'(\eta, t) z'(\eta, t) - \alpha f(\eta, t) z''(\eta, t) \\ & z(0, t) = 0 \quad , z'(0, t) = 0 \quad \text{and} \quad z''(0, t) = 1 \end{aligned} \tag{7}$$

Newton's method therefore requires that two initial-value problems be solved for each iteration, equations (3) and (7). Then from equation (5)

$$t_k = t_{k-1} - \frac{f'(\eta_\infty, t_{k-1}) - 1}{z'(\eta_\infty, t_{k-1})} \tag{8}$$

The convergence of the shooting method based on Newton's iteration is discussed in Burden and Faires (1993) and Granas *et al.* (1979).

Equations (3) and (7) can be written as the following system of initial value problems

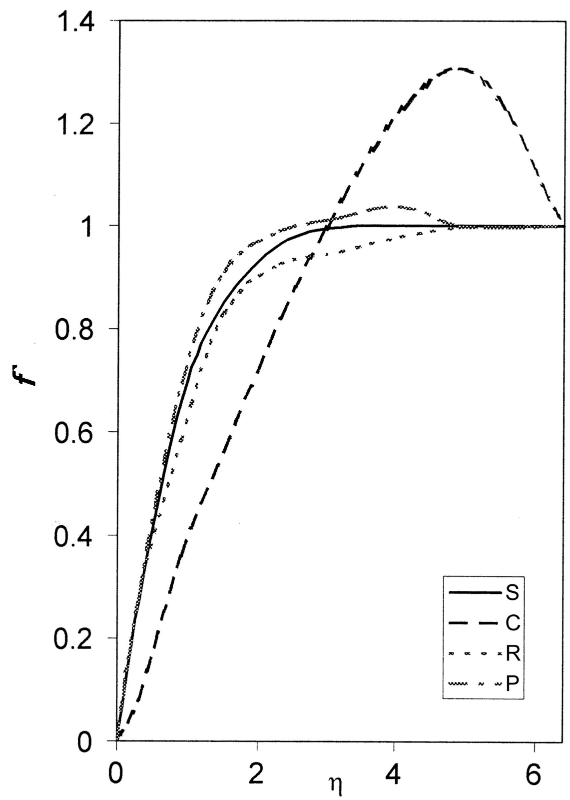

Figure 4. Comparison between spline approximation method (S), Chebyshev spectral Ritz (C) method, Rosenhead (R) and Pseudo-spectral (P) methods, $\beta = 0.5$ and $\alpha = 1$, $\eta_\infty = 6.4$

$$
\begin{aligned}
y_1' &= y_2 \qquad , \quad y_1 = f(\eta, t)\\
y_2' &= y_3\\
y_3' &= -\alpha y_1 y_3 + \beta\left(y_2^2 - 1\right)\\
y_4' &= y_5 \qquad , \quad y_4 = z(\eta, t)\\
y_5' &= y_6\\
y_6' &= -\alpha y_3 y_4 + 2\beta y_2 y_5 - \alpha y_1 y_6
\end{aligned}
$$

$$y_1(0,t) = 0,\ \ y_2(0,t) = 0,\ \ y_3(0,t) = t,\ \ y_4(0,t) = 0,\ \ y_5(0,t) = 0,\ y_6(0,t) = 1$$

which can be set in the form

$$y'(\eta, t) = F(\eta, t, y),\ y(0,t) = y_0\ ,\ \ \eta \in [0, \eta_\infty] \tag{9}$$

Numerical results

We consider the numerical example given in equation (3a) with the boundary conditions (equation (3b)). In Table I we present the resulting wall shear stress f''_w from the proposed spline method. In Table II we give a comparison of the resulting wall shear stress f''_w with those obtained by El-Gindy *et al.* (1995), El-Hawary (1990), Rosenhead (1963), Wadia and Payne (1981), White (1974) and Beckett (1983).

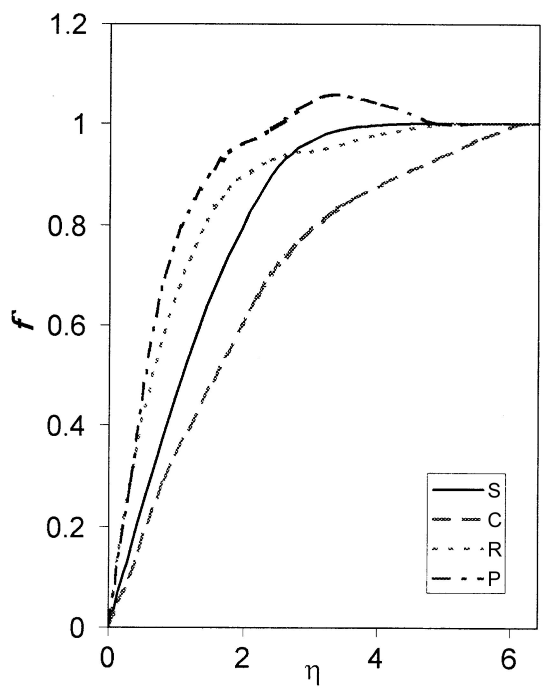

Figure 5.
Comparison between spline approximation method (S), Chebyshev spectral Ritz (C) method, Rosenhead (R) and pseudo-spectral (P) methods, $\beta = 0.0$ and $\alpha = 1$, $\eta_\infty = 6.4$

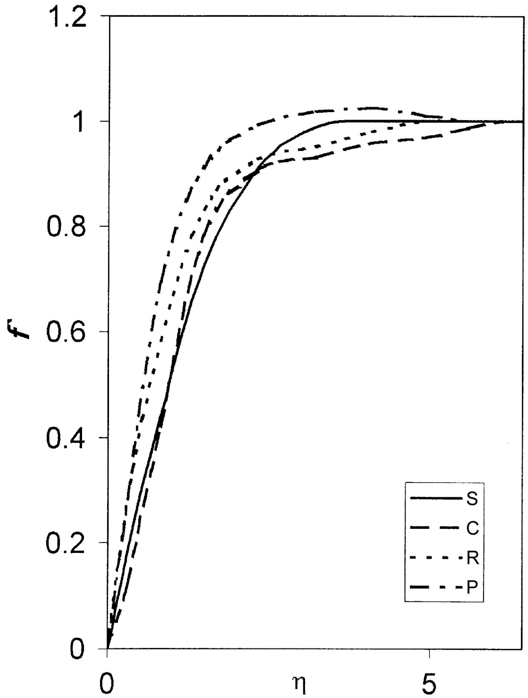

Figure 6.
Comparison between spline approximation method (S), Chebyshev spectral Ritz (C) method, Rosenhead (R) and Pseudo-spectral (P) methods, $\beta = 1.0$ and $\alpha = 1$, $\eta_\infty = 6.4$

Figures 1-3 are obtained from the proposed spline method. These Figures represent the velocity f' for different values of β, α and η_∞. Figures 4-6 present a comparison between the ultraspherical integral method and the data obtained from the Chebyshev spectral-Ritz method (El-Gindy *et al.*, 1995), Rosenhead (Thomas and Harris, 1984) and the pseudo-spectral method (Wadia and Payne, 1981) with different values of β.

Conclusions

The tables and the figures given previously show that the suggested technique is quite reliable. There has been good agreement with results obtained by El- Gindy *et al.* (1995), El-Hawary (1990), Rosenhead (1963), Wadia and Payne (1981), White (1974) and Beckett (1983) for several values of β, α and η_∞.

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